

Tensoring with infinite-dimensional modules in \mathcal{O}_0

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Abstract

We show that the principal block \mathcal{O}_0 of the BGG category \mathcal{O} for a semisimple Lie algebra \mathfrak{g} acts faithfully on itself via exact endofunctors which preserve tilting modules, via right exact endofunctors which preserve projective modules and via left exact endofunctors which preserve injective modules. The origin of all these functors is tensoring with arbitrary (not necessarily finite-dimensional) modules in the category \mathcal{O} . We study such functors, describe their adjoints and show that they give rise to a natural (co)monad structure on \mathcal{O}_0 . Furthermore, all this generalises to parabolic subcategories of \mathcal{O}_0 . As an example, we present some explicit computations for the algebra \mathfrak{sl}_3 .

1 Introduction

When studying the category \mathcal{O} for a semisimple Lie algebra \mathfrak{g} , tensoring with finite dimensional \mathfrak{g} -modules gives rise to a class of functors of high importance, the so called projective functors. These functors were classified in [BG] and include the “translation functors”, [J], which can be used to prove equivalences of certain subcategories of \mathcal{O} .

In the following we study tensoring with arbitrary (not necessarily finite dimensional) modules in \mathcal{O} . There is an immediate obstacle, namely the fact that, in general, the result is no longer finitely generated (in other words, such functors do not preserve \mathcal{O}). This can be remedied by projecting onto a fixed block of the category \mathcal{O} . In particular, by composing with projection to the principal block \mathcal{O}_0 , we obtain a faithful, exact functor $G: M \mapsto G_M := M \otimes _ \downarrow_0$ from \mathcal{O}_0 to the category $\text{End}(\mathcal{O}_0)$ of endofunctors on \mathcal{O}_0 . By, defining F_M and H_M to be the left and right adjoints of G_M , we obtain a right exact contravariant functor $F: M \mapsto F_M$ and a left exact contravariant functor $H: M \mapsto H_M$ from \mathcal{O} to $\text{End}(\mathcal{O}_0)$.

In Section 2 we introduce the required notions and notation, and provide a setting for studying the tensor product of arbitrary modules in \mathcal{O} . In Section 3 we define the three functors, and determine some of their properties. The main properties are given by Theorem 3.1, which shows that F_M preserves projectives, G_M preserves tilting modules, and H_M preserves injectives, for any $M \in \mathcal{O}_0$. In Section 4 we show that the particular functors $G_{\Delta(0)}$ and $G_{\nabla(0)}$ have natural comonad and monad structures, respectively. In Section 5 we show how the results from the previous section generalize to parabolic subcategories of \mathcal{O} . Finally, in Section 6 we compute the ‘multiplication tables’ $G_M N$ and $F_M N$ for the case $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, where M and N run over the simple modules in \mathcal{O}_0 .

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2 Notation and preliminaries

For any Lie algebra \mathfrak{a} , we let $\mathcal{U}(\mathfrak{a})$ denote its universal enveloping algebra. Fix \mathfrak{g} to be a finite dimensional complex semisimple Lie algebra, with a chosen triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ denote the Borel subalgebra, and let R denote the corresponding root system, with positive roots R_+ , negative roots R_- , and basis Π . Let \mathcal{O} denote the corresponding BGG-category (see [BGG] for details), which can be defined as the full subcategory of the category of \mathfrak{g} -modules consisting of weight modules that are finitely generated as $\mathcal{U}(\mathfrak{n}_-)$ -modules

For a weight module M , we denote by M_λ the subspace of M of weight $\lambda \in \mathfrak{h}^*$, and by $\text{Supp } M := \{ \lambda \in \mathfrak{h}^* \mid M_\lambda \neq \{0\} \}$ the support of M . For a weight vector $v \in M$, we denote by $w(v)$ the weight of v , i.e. $v \in M_{w(v)}$. Let \mathbb{N}_0 denote the non-negative integers, and let \leq denote the natural partial order on \mathfrak{h}^* , i.e. $\lambda \leq \mu$ if and only if $\lambda - \mu \in \mathbb{N}_0 R_-$.

Given an anti-automorphism $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$ of \mathfrak{g} we define the corresponding restricted duality d on the category of weight \mathfrak{g} -modules as follows. For a weight \mathfrak{g} -module M , let

$$dM := \bigoplus_{\lambda \in \mathfrak{h}^*} \text{Hom}_{\mathbb{C}}(M_\lambda, \mathbb{C}),$$

with the action of \mathfrak{g} given by

$$(xf)(m) := f(\theta(x)m),$$

for $x \in \mathfrak{g}$, $f \in dM$ and $m \in M$.

We will use two different restricted dualities on weight \mathfrak{g} -modules: the duality given by the anti-automorphism $\mathfrak{g} \rightarrow \mathfrak{g}$, $x \mapsto -x$, which we will denote by M^* , and the duality given by the Chevalley anti-automorphism, which we will denote by M^\star . Note that $\text{Supp } M^\star = \text{Supp } M$, and thus \star preserves the category \mathcal{O} , whereas $\text{Supp } M^* = -\text{Supp } M$. ‘The dual of M ’, ‘ M is self-dual’ and similar statements will, unless otherwise stated, refer to the \star -duality.

Since \mathcal{O} is not closed under tensor products (e.g. the tensor product of two Verma modules is never finitely generated and hence does not belong to \mathcal{O}), it would be convenient to define the ‘enlarged’ category $\tilde{\mathcal{O}}$, as the full subcategory of weight \mathfrak{g} -modules M having the properties

(OT1) there are weights $\lambda_1, \dots, \lambda_k \in \mathfrak{h}^*$ with

$$\text{Supp } M \subseteq \bigcup_{i=1}^k (\lambda_i + \mathbb{N}_0 R_-),$$

(OT2) $\dim_{\mathbb{C}} M_\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$.

Lemma 2.1. *The category $\tilde{\mathcal{O}}$ is closed under tensor products.*

Proof. Let $M, N \in \tilde{\mathcal{O}}$. Then $M \otimes N$ is a weight module, and since

$$\text{Supp}(M \otimes N) = \text{Supp } M + \text{Supp } N, \quad (1)$$

it is easy to see that the property (OT1) is preserved under tensor products. Also,

$$\dim(M \otimes N)_\lambda = \sum_{\substack{\mu \in \text{Supp } M, \\ \nu \in \text{Supp } N, \\ \mu + \nu = \lambda}} \dim M_\mu \cdot \dim N_\nu. \quad (2)$$

By (OT1) the set of pairs $\mu \in \text{Supp } M$, $\nu \in \text{Supp } N$ with $\mu + \nu = \lambda$ is finite for any $\lambda \in \mathfrak{h}^*$. By (OT2) we have that $\dim M_\mu < \infty$ and $\dim N_\nu < \infty$ for any μ and ν , so it follows that the right hand side of (2) is finite, i.e. $\dim(M \otimes N)_\lambda < \infty$. \square

Lemma 2.2. *The duality \star commutes with tensor products in $\tilde{\mathcal{O}}$, that is*

$$(M \otimes N)^\star \cong M^\star \otimes N^\star,$$

natural in M and N .

Proof. For $f^\star \in M^\star$ and $g^\star \in N^\star$, let $\psi(f^\star \otimes g^\star) \in (M \otimes N)^\star$ be defined by

$$\psi(f^\star \otimes g^\star)(m \otimes n) := f^\star(m)g^\star(n),$$

for $m \in M$ and $n \in N$, and extended bilinearly to a map $M^* \otimes N^* \rightarrow (M \otimes N)^*$. Straightforward verification shows that this is a homomorphism, natural in both M and N . Let $m_1, m_2, \dots \in M$ and $n_1, n_2, \dots \in N$ be bases of weight vectors, and let $m_1^*, m_2^*, \dots \in M^*$ and $n_1^*, n_2^*, \dots \in N^*$ be the corresponding dual bases. Then we have that $\{m_i \otimes n_j \mid i, j = 1, 2, \dots\}$ is a basis of $M \otimes N$, with the dual basis $\{(m_i \otimes n_j)^* \mid i, j = 1, 2, \dots\}$. Furthermore,

$$\begin{aligned} \psi(m_i^* \otimes n_j^*)(m_k \otimes n_l) &= m_i^*(m_k)n_j^*(n_l) \\ &= \delta_{ik}\delta_{jl} \\ &= (m_i \otimes n_j)^*(m_k \otimes n_l), \end{aligned}$$

i.e. $\psi(m_i^* \otimes n_j^*) = (m_i \otimes n_j)^*$, so ψ is indeed an isomorphism. \square

Note that \mathcal{O} is the full subcategory of $\tilde{\mathcal{O}}$ consisting of finitely generated modules, and in particular the simple objects of $\tilde{\mathcal{O}}$ and \mathcal{O} coincide. For $\lambda \in \mathfrak{h}^*$, let $L(\lambda)$ denote the simple highest weight module with highest weight λ , and let $P(\lambda)$ denote the projective cover of $L(\lambda)$.

Lemma 2.3. *All modules $M \in \tilde{\mathcal{O}}$ admit a (possibly infinite) composition series. Furthermore, for each $\lambda \in \mathfrak{h}^*$, the number $[M : L(\lambda)]$ of occurrences of $L(\lambda)$ as a composition factor in a composition series is finite and independent of the choice of composition series.*

Proof. Let $M \in \tilde{\mathcal{O}}$, and let $m_1, m_2, m_3, \dots \in M$ be a basis of weight vectors such that $w(m_i) \leq w(m_j)$ implies that $j \leq i$. Such a basis exists due to (OT1) and (OT2). For $i \in \mathbb{N}_0$, let $M^{(i)}$ denote the submodule of M generated by $\{m_j \mid j \leq i\}$. We thus obtain a series of finitely generated modules

$$\{0\} = M^{(0)} \subseteq M^{(1)} \subseteq M^{(2)} \subseteq M^{(3)} \subseteq \dots,$$

which, since the m_i 's constitute a basis of M , converge to M , i.e.

$$\bigcup_{i=0}^{\infty} M^{(i)} = M.$$

Since the $M^{(i)}$'s are finitely generated, $M^{(i)} \in \mathcal{O}$ for all $i \in \mathbb{N}_0$. Thus, since all objects in \mathcal{O} have finite length, this series can be refined to a composition series.

Now, consider any composition series $(M^{(i)})$ of M , let $\lambda \in \mathfrak{h}^*$ be any weight of M , and let N denote the submodule of M generated by the weight space M_λ . Since $\dim M_\lambda < \infty$ there exists an index $k \in \mathbb{N}$

such that $M_\lambda \subseteq M^{(k)}$, and in particular such that N is a submodule of $M^{(k)}$. Then $(M^{(i)}/N)_\lambda = \{0\}$ for all $i \geq k$, so

$$[(M^{(i)}/N) : L(\lambda)] = 0$$

for all $i \geq k$, and thus

$$[M^{(i)} : L(\lambda)] = [N : L(\lambda)]$$

for all $i \geq k$. As $N \in \mathcal{O}$, we get that $[M : L(\lambda)] = [N : L(\lambda)]$ is finite and independent of the choice of composition series. \square

Recall that \mathcal{O} has a block decomposition

$$\mathcal{O} = \bigoplus_{\chi \in \mathcal{Z}(\mathfrak{g})^*} \mathcal{O}_\chi,$$

where $\mathcal{Z}(\mathfrak{g})$ denotes the centre of \mathfrak{g} and \mathcal{O}_χ denotes the full subcategory of \mathcal{O} consisting of modules M such that for all $z \in \mathcal{Z}(\mathfrak{g})$, M is annihilated by some power of $(z - \chi(z))$. Hence, each module $M \in \mathcal{O}$ decomposes into direct sum

$$M = \bigoplus_{\chi \in \mathcal{Z}(\mathfrak{g})^*} M_\chi, \tag{3}$$

where $M_\chi \in \mathcal{O}_\chi$ and $M_\chi \neq \{0\}$ for only finitely many χ .

From Lemma 2.3 it follows that we get a similar block decomposition for $\tilde{\mathcal{O}}$, where each module $M \in \tilde{\mathcal{O}}$ decomposes as in (3), but with possibly countably many non-zero summands (and with some restrictions on the weight spaces of the non-zero summands). This is similar to the situation for \mathcal{O} -like categories over a Kac-Moody algebra, see for example [N1, R-CW]. More precisely, we have the following.

Lemma 2.4. *For all $M \in \tilde{\mathcal{O}}$ and all $\chi \in \mathcal{Z}(\mathfrak{g})^*$ there are unique modules (up to isomorphism) $M_1 \in \mathcal{O}_\chi$, $M_2 \in \tilde{\mathcal{O}}$, with $[M_2 : L(\mu)] = 0$ for all $\mu \in \mathfrak{h}^*$ with $L(\mu) \in \mathcal{O}_\chi$, such that*

$$M \cong M_1 \oplus M_2.$$

Proof. Recall that, for two \mathfrak{g} -modules K and N , the trace $\text{Tr}_K N$ is defined as the sum of images of all homomorphisms from K to N . Now, let

$$M_1 := \sum_{\substack{\lambda \in \mathfrak{h}^* \\ P(\lambda) \in \mathcal{O}_\chi}} \text{Tr}_{P(\lambda)} M,$$

and

$$M_2 := \sum_{\substack{\lambda \in \mathfrak{h}^*, \\ P(\lambda) \notin \mathcal{O}_\chi}} \text{Tr}_{P(\lambda)} M.$$

As \mathcal{O} has enough projectives, from the proof of Lemma 2.3 it follows that $M = M_1 + M_2$. Since the central characters occuring in M_2 are different from χ , this sum must be direct. \square

For each $\chi \in \mathcal{Z}(\mathfrak{g})^*$ we thus obtain an exact projection functor $_ \downarrow_\chi : \tilde{\mathcal{O}} \rightarrow \mathcal{O}_\chi$, such that

$$M = \bigoplus_{\chi \in \mathcal{Z}(\mathfrak{g})^*} M \downarrow_\chi \quad (4)$$

for any $M \in \tilde{\mathcal{O}}$.

Lemma 2.5. *The tensor product commutes with infinite direct sums in $\tilde{\mathcal{O}}$.*

Proof. Let $N, M_1, M_2, \dots \in \tilde{\mathcal{O}}$ with

$$\bigoplus_{i=1}^{\infty} M_i \in \tilde{\mathcal{O}},$$

let $n_1, n_2, \dots \in N$ be a basis of N and let $m_1^{(i)}, m_2^{(i)}, \dots \in M_i$ be a basis of M_i for each $i \in \mathbb{N}$. Then it is immediate that

$$\{ m_j^{(i)} \otimes n_k \mid i, j, k \in \mathbb{N} \}$$

constitute a basis of both

$$(M_1 \oplus M_2 \oplus \dots) \otimes N$$

and

$$(M_1 \otimes N) \oplus (M_2 \otimes N) \oplus \dots,$$

giving the required isomorphism. \square

For $\lambda \in \mathfrak{h}^*$, we denote by $\Delta(\lambda)$ the corresponding Verma module with highest weight λ , and $\nabla(\lambda) := \Delta(\lambda)^*$ the corresponding dual Verma module. Let $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ denote the categories of modules $M \in \mathcal{O}$ having a Verma filtration and dual Verma filtration, respectively, and let $\mathcal{T} = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ denote the category of tilting modules (see [R] for more details). Let $\tilde{\mathcal{F}}(\Delta)$, $\tilde{\mathcal{F}}(\nabla)$ and $\tilde{\mathcal{T}}$ denote the corresponding categories for $\tilde{\mathcal{O}}$. As \star commutes with direct sums, the decomposition (4) implies that $M \in \tilde{\mathcal{F}}(\Delta)$ if and only if $M^* \in \tilde{\mathcal{F}}(\nabla)$.

Note also that $\mathcal{F}(\Delta)$ and $\tilde{\mathcal{F}}(\Delta)$ can be characterised as the objects in \mathcal{O} and $\tilde{\mathcal{O}}$ respectively which are free as $\mathcal{U}(\mathfrak{n}_-)$ -modules.

Similar to the situation in \mathcal{O} , we have the following result for $\tilde{\mathcal{O}}$ concerning tensor products involving (dual) Verma modules and tilting modules.

Proposition 2.6. *For any $M \in \tilde{\mathcal{O}}$, $N \in \tilde{\mathcal{F}}(\Delta)$, $K \in \tilde{\mathcal{F}}(\nabla)$ and $T \in \tilde{\mathcal{T}}$ we have $M \otimes N \in \tilde{\mathcal{F}}(\Delta)$, $M \otimes K \in \tilde{\mathcal{F}}(\nabla)$ and $M \otimes T \in \tilde{\mathcal{T}}$.*

Proof. To show $M \otimes N \in \tilde{\mathcal{F}}(\Delta)$, it suffices to show that $M \otimes N \in \tilde{\mathcal{F}}(\Delta)$ for any $N \in \mathcal{F}(\Delta)$, since the general case then follows from the fact that any module in $\tilde{\mathcal{F}}(\Delta)$ decomposes into a direct sum of modules in $\mathcal{F}(\Delta)$. Let $m_1, m_2, \dots \in M$ be a basis of M constructed as in the proof of Lemma 2.3 and let $v_1, \dots, v_k \in N$ be a basis of N as a $\mathcal{U}(\mathfrak{n}_-)$ -module consisting of weight vectors. We will now show that $M \otimes N$ is $\mathcal{U}(\mathfrak{n}_-)$ -free with the basis $B := \{m_i \otimes v_j \mid i \in \mathbb{N}, 1 \leq j \leq k\}$.

We start by showing that B generates $M \otimes N$ as a $\mathcal{U}(\mathfrak{n}_-)$ -module. A set that certainly generates $M \otimes N$ over $\mathcal{U}(\mathfrak{n}_-)$ is

$$\bar{B} := \{m_i \otimes (uv_j) \mid i \in \mathbb{N}, u \in \mathcal{U}(\mathfrak{n}_-), 1 \leq j \leq k\},$$

since $\{m_1, m_2, \dots\}$ is a basis of M and

$$\sum_{j=1}^k \{uv_j \mid u \in \mathcal{U}(\mathfrak{n}_-)\} = N.$$

We will show that \bar{B} is a subset of the set generated by B by induction on the degree of u . So, consider an element $m_i \otimes (uv_j) \in \bar{B}$. If u has degree 0, then u is a scalar, so $m_i \otimes (uv_j) = u(m_i \otimes v_j)$ is in the set generated by B . Now assume u has degree $d \geq 1$. Then

$$m_i \otimes (uv_j) = u(m_i \otimes v_j) + \sum_l (u'_l m_i) \otimes (u''_l v_j),$$

for some elements $u'_l, u''_l \in \mathcal{U}(\mathfrak{n}_-)$ with degree strictly less than d . Since we can rewrite the elements $u'_l m_i$ as linear combinations of m_1, m_2, \dots , the right hand side is in the set generated by B over $\mathcal{U}(\mathfrak{n}_-)$ by the induction hypothesis. Hence B generates \bar{B} as a $\mathcal{U}(\mathfrak{n}_-)$ -module, so B generates $M \otimes N$ as a $\mathcal{U}(\mathfrak{n}_-)$ -module.

To see that $M \otimes N$ is free over B as a $\mathcal{U}(\mathfrak{n}_-)$ -module, let L_l denote the $\mathcal{U}(\mathfrak{n}_-)$ -submodule of $M \otimes N$ generated by

$$\{m_i \otimes v_j \mid i \in \mathbb{N}, i \leq l, 1 \leq j \leq k\},$$

and let \bar{L}_l denote the $\mathcal{U}(\mathfrak{n}_-)$ -submodule of $M \otimes N$ generated by

$$\{m_l \otimes v_j \mid 1 \leq j \leq k\}.$$

By straightforward induction we see that any non-zero element in L_l has a summand of the form $m_i \otimes n$ for some $1 \leq i \leq k$, $n \in N$. On the other hand, no element of \bar{L}_{l+1} has such a summand by the ordering of the m_i 's, and hence we have

$$L_{l+1} = \bar{L}_{l+1} \oplus L_l.$$

Thus

$$M \otimes N = \bigoplus_{l=1}^{\infty} \bar{L}_l$$

as a $\mathcal{U}(\mathfrak{n}_-)$ -module. Finally, we note that \bar{L}_l is $\mathcal{U}(\mathfrak{n}_-)$ -free with the generators

$$\{ m_l \otimes v_j \mid 1 \leq j \leq k \},$$

since $u(m_l \otimes v_j)$ has a summand of the form $m_l \otimes (uv_j)$ for all $u \in \mathcal{U}(\mathfrak{n}_-)$. Hence $M \otimes N$ is $\mathcal{U}(\mathfrak{n}_-)$ -free, i.e. $M \otimes N \in \tilde{\mathcal{F}}(\Delta)$.

To show that $M \otimes K \in \tilde{\mathcal{F}}(\nabla)$, note that since $K^* \in \tilde{\mathcal{F}}(\Delta)$, by the previous paragraph we have $M^* \otimes K^* \in \tilde{\mathcal{F}}(\Delta)$. By Lemma 2.2, \star commutes with tensor products, i.e. $M^* \otimes K^* = (M \otimes K)^*$, and hence $M \otimes K \in \tilde{\mathcal{F}}(\nabla)$.

Finally, since $\tilde{\mathcal{T}} = \tilde{\mathcal{F}}(\Delta) \cap \tilde{\mathcal{F}}(\nabla)$, from the first two statements it follows that $M \otimes T \in \tilde{\mathcal{T}}$ for all $M \in \tilde{\mathcal{O}}$ and $T \in \tilde{\mathcal{T}}$. \square

Corollary 2.7. *For $M \in \tilde{\mathcal{F}}(\Delta)$ and $N \in \tilde{\mathcal{F}}(\nabla)$ we have $M \otimes N \in \tilde{\mathcal{T}}$. Furthermore, if $\lambda_1, \lambda_2, \dots \in \mathfrak{h}^*$ and $\mu_1, \mu_2, \dots \in \mathfrak{h}^*$ are the highest weights, with multiplicities, of the Verma (respectively dual Verma) modules occurring in the Verma and dual Verma filtrations of M and N , then*

$$M \otimes N \cong \bigoplus_{i,j=1}^{\infty} \Delta(\lambda_i) \otimes \nabla(\mu_j).$$

Proof. By Proposition 2.6, $M \otimes N \in \tilde{\mathcal{F}}(\Delta) \cap \tilde{\mathcal{F}}(\nabla) = \tilde{\mathcal{T}}$. Furthermore, $\Delta(\lambda) \otimes \nabla(\mu) \in \tilde{\mathcal{T}}$ for all $\lambda, \mu \in \mathfrak{h}^*$. Since tensoring over a field, the second statement now follows from the fact that tilting modules do not have self-extensions [R, Corollary 3]. \square

Following [F], for $\lambda \in \mathfrak{h}^*$ and any weight module M we define

$$M^{\leq \lambda} := M / M^{\not\leq \lambda},$$

where $M^{\not\leq \lambda}$ is the submodule of M generated by all the weight spaces M_μ with $\mu \not\leq \lambda$.

Lemma 2.8. *The assignment $_{}^{\leq \lambda}: M \mapsto M^{\leq \lambda}$ defines a right exact functor on the category of weight \mathfrak{g} -modules.*

Proof. Let M and N be weight \mathfrak{g} -modules, and let $\varphi : M \rightarrow N$ be a homomorphism. Since homomorphisms preserve weights, the generating set for $M^{\not\leq \lambda}$ maps to the generating set for $N^{\not\leq \lambda}$, and hence $\varphi(M^{\not\leq \lambda}) \subseteq N^{\not\leq \lambda}$. We thus obtain an induced homomorphism

$$\varphi^{\leq \lambda} : M^{\leq \lambda} \rightarrow N^{\leq \lambda}.$$

It is immediate that $(\text{Id}_M)^{\leq \lambda} = \text{Id}_{M^{\leq \lambda}}$ and $(\varphi \circ \psi)^{\leq \lambda} = \varphi^{\leq \lambda} \circ \psi^{\leq \lambda}$, so $\underline{}^{\leq \lambda}$ is indeed a functor.

Now, consider an exact sequence

$$K \xrightarrow{\psi} M \xrightarrow{\varphi} N \rightarrow 0$$

of weight \mathfrak{g} -modules. For any element $n + N^{\not\leq \lambda} \in N^{\leq \lambda}$, there is an element $m \in M$ with $\varphi(m) = n$, so

$$\varphi^{\leq \lambda}(m + M^{\not\leq \lambda}) = n + N^{\not\leq \lambda},$$

and thus $\varphi^{\leq \lambda}$ is surjective. Finally, consider an element

$$m + M^{\not\leq \lambda} \in \ker \varphi^{\leq \lambda},$$

i.e. $\varphi(m + M^{\not\leq \lambda}) \subseteq N^{\not\leq \lambda}$. Since φ is surjective, we have $\varphi(M^{\not\leq \lambda}) = N^{\not\leq \lambda}$, so there is an element $\tilde{m} \in M^{\not\leq \lambda}$ with $\varphi(\tilde{m}) = \varphi(m)$. Now let $m' = m - \tilde{m}$. Since

$$\varphi(m') = \varphi(m) - \varphi(\tilde{m}) = 0$$

we have $m' \in \ker \varphi$, and since $\tilde{m} \in M^{\not\leq \lambda}$ we have

$$m' + M^{\not\leq \lambda} = m + M^{\not\leq \lambda}.$$

By exactness, there is an element $k \in K$ with $\psi(k) = m'$, so

$$\psi^{\leq \lambda}(k + K^{\not\leq \lambda}) = m' + M^{\not\leq \lambda} = m + M^{\not\leq \lambda}.$$

Hence $\text{im } \psi^{\leq \lambda} = \ker \varphi^{\leq \lambda}$, and thus $\underline{}^{\leq \lambda}$ is right exact. \square

Proposition 2.9. *Let M be an $\mathcal{U}(\mathfrak{n}_-)$ -free module, say*

$$M = \bigoplus_{i \in I} \mathcal{U}(\mathfrak{n}_-) v_i$$

as an $\mathcal{U}(\mathfrak{n}_-)$ -module with $\{v_i \mid i \in I\}$ being weight vectors. Then

$$M^{\leq \lambda} \cong \bigoplus_{\substack{i \in I, \\ \mathbf{w}(v_i) \leq \lambda}} \mathcal{U}(\mathfrak{n}_-) v_i.$$

as a $\mathcal{U}(\mathfrak{n}_-)$ -module.

Proof. We claim that

$$M^{\not\leq \lambda} = \sum_{\substack{i \in I, \\ w(v_i) \not\leq \lambda}} \mathcal{U}(\mathfrak{n}_-) v_i.$$

To show this, let N denote the set on the right hand side. We need to show that N is indeed a submodule of M , i.e. closed under the action of $\mathcal{U}(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt Theorem we know that $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_-) \mathcal{U}(\mathfrak{b})$, so

$$\begin{aligned} \mathcal{U}(\mathfrak{g})N &= \sum_{\substack{i \in I, \\ w(v_i) \not\leq \lambda}} \mathcal{U}(\mathfrak{g}) \mathcal{U}(\mathfrak{n}_-) v_i \\ &= \sum_{\substack{i \in I, \\ w(v_i) \not\leq \lambda}} \mathcal{U}(\mathfrak{g}) v_i \\ &= \sum_{\substack{i \in I, \\ w(v_i) \not\leq \lambda}} \mathcal{U}(\mathfrak{n}_-) \mathcal{U}(\mathfrak{b}) v_i \\ &\stackrel{(*)}{=} \sum_{\substack{i \in I, \\ w(v_i) \not\leq \lambda}} \mathcal{U}(\mathfrak{n}_-) v_i \\ &= N, \end{aligned}$$

where $(*)$ holds since if $w(v_i) \not\leq \lambda$, then $\mu \not\leq \lambda$ for any $\mu \in \text{Supp}(\mathcal{U}(\mathfrak{b})v_i)$. Thus, as a $\mathcal{U}(\mathfrak{n}_-)$ -module, we have

$$M^{\not\leq \lambda} = \bigoplus_{\substack{i \in I, \\ w(v_i) \not\leq \lambda}} \mathcal{U}(\mathfrak{n}_-) v_i,$$

and hence we get that

$$M^{\leq \lambda} \cong \bigoplus_{\substack{i \in I, \\ w(v_i) \leq \lambda}} \mathcal{U}(\mathfrak{n}_-) v_i.$$

as a $\mathcal{U}(\mathfrak{n}_-)$ -module. □

Proposition 2.10. *For any $M \in \mathcal{F}(\Delta)$, $N \in \tilde{\mathcal{O}}$ and $\lambda \in \mathfrak{h}^*$ we have that $(M \otimes N^*)^{\leq \lambda} \in \mathcal{F}(\Delta)$.*

Proof. Let $m_1, \dots, m_k \in M$ be a basis of M as a $\mathcal{U}(\mathfrak{n}_-)$ -module consisting of weight vectors, and let $n_1, n_2, \dots \in N$ be a basis of N constructed as in the proof of Lemma 2.3. By an argument completely analogous to the case where N is finite dimensional (see for instance

the proof of Theorem 2.2 in [J]), it follows that $M \otimes N^*$ is $\mathcal{U}(\mathfrak{n}_-)$ -free over the set

$$B = \{ m_i \otimes n_j^* \mid 1 \leq i \leq k, j \in \mathbb{N} \}.$$

By Proposition 2.9 it follows that $(M \otimes N^*)^{\leq \lambda}$ is $\mathcal{U}(\mathfrak{n}_-)$ -free, with a $\mathcal{U}(\mathfrak{n}_-)$ -basis consisting of the vectors in B satisfying $w(m_i \otimes n_j^*) \leq \lambda$. Since $N \in \tilde{\mathcal{O}}$, the number of such vectors is finite, and hence $(M \otimes N^*)^{\leq \lambda} \in \mathcal{O}$, i.e. $(M \otimes N^*)^{\leq \lambda} \in \mathcal{F}(\Delta)$. \square

Corollary 2.11. *For each $M \in \mathcal{O}$, $N \in \tilde{\mathcal{O}}$ and $\lambda \in \mathfrak{h}^*$ we have*

$$(M \otimes N^*)^{\leq \lambda} \in \mathcal{O}.$$

Proof. Let $P \in \mathcal{O}$ be the projective cover of M . As $_{}^{\leq \lambda}$ is right exact, it suffices to prove that $(P \otimes N^*)^{\leq \lambda} \in \mathcal{O}$. But this follows from Proposition 2.10, since every projective in \mathcal{O} has a Verma flag. \square

3 The functors

We now restrict our attention to the principal block \mathcal{O}_0 , i.e. the indecomposable block containing the trivial module $L(0)$. Let $\text{PFun}(\mathcal{O}_0)$, $\text{TFun}(\mathcal{O}_0)$ and $\text{IFun}(\mathcal{O}_0)$ denote the categories of endofunctors on \mathcal{O}_0 which preserve the additive subcategories of projective, tilting and injective modules, respectively. Furthermore, let $\mathcal{F}_0(\Delta) = \mathcal{F}(\Delta) \cap \mathcal{O}_0$, and define $\mathcal{F}_0(\nabla)$ and \mathcal{T}_0 similarly. This section will be devoted to proving the following theorem, the main result of this paper, along with some of its consequences.

Theorem 3.1. *There exist faithful functors*

$$F: \mathcal{O}_0 \hookrightarrow \text{PFun}(\mathcal{O}_0)^{op}, M \mapsto F_M,$$

$$G: \mathcal{O}_0 \hookrightarrow \text{TFun}(\mathcal{O}_0), M \mapsto G_M,$$

$$H: \mathcal{O}_0 \hookrightarrow \text{IFun}(\mathcal{O}_0)^{op}, M \mapsto H_M,$$

all three satisfying $X_M \cong X_N$ if and only if $M \cong N$ (where $X = F, G, H$).

For $M \in \mathcal{O}_0$, we define the functor $G_M: \mathcal{O}_0 \rightarrow \mathcal{O}_0$ by

$$G_M N := (M \otimes N) \downarrow_0$$

on objects, and

$$G_M \varphi: G_M K \rightarrow G_M L,$$

$$G_M \varphi = (\text{Id}_M \otimes \varphi) \downarrow_0 := \pi_{G_M L} \circ (\text{Id}_M \otimes \varphi) \circ \iota_{G_M K},$$

on morphisms $\varphi: K \rightarrow L$, where $\pi_{G_M L}: M \otimes L \rightarrow (M \otimes L)_{\downarrow 0}$ and $\iota_{G_M K}: (M \otimes K)_{\downarrow 0} \hookrightarrow M \otimes K$ denote the natural projection and inclusion. This defines G_M as an endofunctor on \mathcal{O}_0 .

Remark 3.2. By central character considerations (i.e. from the fact that $G_M L \in \mathcal{O}_0$), it follows that $\pi_{G_M L} \circ (\text{Id}_M \otimes \varphi)$ factors through $G_M \varphi$, i.e. the diagram

$$\begin{array}{ccc} M \otimes K & \xrightarrow{\text{Id}_M \otimes \varphi} & M \otimes L \\ \pi_{G_M K} \downarrow & & \downarrow \pi_{G_M L} \\ G_M K & \xrightarrow{G_M \varphi} & G_M L \end{array}$$

commutes.

For a homomorphism $\varphi: M \rightarrow N$ between to objects $M, N \in \mathcal{O}_0$, we define the corresponding natural transformation $G_\varphi: G_M \rightarrow G_N$ by

$$\begin{aligned} G_\varphi K &: G_M K \rightarrow G_N K, \\ G_\varphi K &:= (\varphi \otimes \text{Id}_K)_{\downarrow 0}, \end{aligned}$$

for $K \in \mathcal{O}_0$. This defines G as a functor from the category \mathcal{O}_0 to the category of endofunctors on \mathcal{O}_0 .

Since both $M \otimes _$ and $_ \downarrow_0$ are exact (as the tensor product is over a field), it follows that G_M is exact. Recall that the category \mathcal{O}_0 is equivalent to $A\text{-mod}$, the category of A -modules, for some finite dimensional algebra A (see [BGG]). Hence G_M can be seen as an exact functor on $A\text{-mod}$, and in particular G_M is right exact on $A\text{-mod}$. By abstract theory (e.g. Theorem 2.3, [B]), G_M is naturally isomorphic to a functor on the form $\overline{M} \otimes_A _$ for some A -bimodule \overline{M} . We define

$$H_M := \text{Hom}_A(\overline{M}, _),$$

the right adjoint of G_M . The dual \star is a self-adjoint contravariant endofunctor on \mathcal{O}_0 , so for any modules $K, L \in \mathcal{O}_0$ we have the following natural isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{O}_0}(L, (G_M \star K^\star)^\star) &\cong \text{Hom}_{\mathcal{O}_0}(G_M \star K^\star, L^\star) \\ &\cong \text{Hom}_{\mathcal{O}_0}(K^\star, H_M \star L^\star) \\ &\cong \text{Hom}_{\mathcal{O}_0}((H_M \star L^\star)^\star, K). \end{aligned}$$

Furthermore, since \star commutes with direct sums and tensor products, we see that

$$(G_M \star K^\star)^\star = ((M^\star \otimes K^\star)_{\downarrow 0})^\star = (M \otimes K)_{\downarrow 0} = G_M K.$$

Thus $\star \circ H_{M^\star} \circ \star$ is the left adjoint of G_M , and we define

$$F_M := \star \circ H_{M^\star} \circ \star. \quad (5)$$

Proposition 3.3. *For any $M \in \mathcal{O}_0$ we have that $F_M \in \text{PFun}(\mathcal{O}_0)$, $G_M \in \text{TFun}(\mathcal{O}_0)$ and $H_M \in \text{IFun}(\mathcal{O}_0)$.*

Proof. That $G_M \in \text{TFun}(\mathcal{O}_0)$ follows from Proposition 2.6. Assume that $P \in \mathcal{O}_0$ is projective, i.e. the functor $\text{Hom}(P, _)$ is exact. We need to show that $F_M P$ is projective, i.e. that $\text{Hom}(F_M P, _)$ is exact. But

$$\text{Hom}(F_M P, _) \cong \text{Hom}(P, G_M _),$$

and the right hand side is the composition of two exact functors, so it is exact. The statement $H_M \in \text{IFun}(\mathcal{O}_0)$ follows by duality. \square

Theorem 3.4. *The left adjoint F_M of G_M is given by*

$$F_M N = (M^\star \otimes N)^{\leq 0} \downarrow_0,$$

and the right adjoint H_M by

$$H_M = \star \circ F_{M^\star} \circ \star.$$

Proof. The second statement follows immediately from the definition (5). The proof of the first assertion is a slight variation of the proof of Proposition 5.1 in [F], also due to Fiebig. We begin by showing that we have a natural isomorphism

$$\text{Hom}_{\mathfrak{g}}(M^\star \otimes K, L) \cong \text{Hom}_{\mathfrak{g}}(K, M \otimes L)$$

for all $K, L, M \in \mathcal{O}_0$. Let $m_1, m_2, \dots \in M$ be a basis consisting of weight vectors, and let $m_1^\star, m_2^\star, \dots \in M^\star$ denote the corresponding dual basis. For $f \in \text{Hom}_{\mathfrak{g}}(M^\star \otimes K, L)$, define $\hat{f} \in \text{Hom}_{\mathfrak{g}}(K, M \otimes L)$ by

$$\hat{f}(k) := \sum_i m_i \otimes f(m_i^\star \otimes k).$$

Since $\text{Supp } L \leq 0$, we see that the sum on the right hand side is finite, since $f(m_i^\star \otimes k) = 0$ for all i with

$$\text{w}(m_i^\star \otimes k) \not\leq 0.$$

For $g \in \text{Hom}_{\mathfrak{g}}(K, M \otimes L)$, define $\tilde{g} \in \text{Hom}_{\mathfrak{g}}(M^\star \otimes K, L)$ by

$$\tilde{g}(m_i^\star \otimes k) := \sum_j m_i^\star(m_j) \cdot l_j,$$

where $g(k) = \sum_j m_j \otimes l_j$ for some weight vectors $l_j \in L$, with $l_j = 0$ for almost all j . The maps $\hat{\cdot}$ and $\tilde{\cdot}$ are indeed inverse to each other, since

$$\tilde{\hat{f}}(m_i^* \otimes k) = \sum_j m_i^*(m_j) \cdot f(m_j^* \otimes k) = f(m_i^* \otimes k),$$

and

$$\hat{\tilde{g}}(k) = \sum_i m_i \otimes \tilde{g}(m_i^* \otimes k) = \sum_{i,j} m_i \otimes (m_i^*(m_j) \cdot l_j) = \sum_i m_i \otimes l_i = g(k),$$

where again $g(k) = \sum_j m_j \otimes l_j$. Hence

$$\mathrm{Hom}_{\mathfrak{g}}(M^* \otimes K, L) \cong \mathrm{Hom}_{\mathfrak{g}}(K, M \otimes L),$$

as claimed.

As we saw above, any element $f \in \mathrm{Hom}_{\mathfrak{g}}(M^* \otimes K, L)$ is zero on $(M^* \otimes K)^{\not\leq 0}$, and hence f factors uniquely through $(M^* \otimes K)^{\leq 0}$, so

$$\mathrm{Hom}_{\mathfrak{g}}((M^* \otimes K)^{\leq 0}, L) \cong \mathrm{Hom}_{\mathfrak{g}}(M^* \otimes K, L).$$

Also, since $L \in \mathcal{O}_0$, any element in $\mathrm{Hom}_{\mathfrak{g}}((M^* \otimes K)^{\leq 0}, L)$ is zero on any block of $(M^* \otimes K)^{\leq 0}$ outside of \mathcal{O}_0 , so

$$\mathrm{Hom}_{\mathfrak{g}}((M^* \otimes K)^{\leq 0}, L) \cong \mathrm{Hom}_{\mathfrak{g}}((M^* \otimes K)^{\leq 0} \downarrow_0, L).$$

Similarly, since $K \in \mathcal{O}_0$ we have

$$\mathrm{Hom}_{\mathfrak{g}}(K, M \otimes L) \cong \mathrm{Hom}_{\mathfrak{g}}(K, M \otimes L \downarrow_0).$$

Thus we have obtained a chain of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(F_M K, L) &= \mathrm{Hom}_{\mathfrak{g}}((M^* \otimes K)^{\leq 0} \downarrow_0, L) \cong \mathrm{Hom}_{\mathfrak{g}}((M^* \otimes K)^{\leq 0}, L) \\ &\cong \mathrm{Hom}_{\mathfrak{g}}(M^* \otimes K, L) \cong \mathrm{Hom}_{\mathfrak{g}}(K, M \otimes L) \\ &\cong \mathrm{Hom}_{\mathfrak{g}}(K, M \otimes L \downarrow_0) = \mathrm{Hom}_{\mathfrak{g}}(K, G_M L). \end{aligned}$$

□

Corollary 3.5. *F and H are contravariant functors, right and left exact respectively, from the category \mathcal{O}_0 to the category of endofunctors on \mathcal{O}_0 .*

Proof. For $M, N \in \mathcal{O}_0$, we have by Theorem 3.4 that

$$F_M N = (M^* \otimes N)^{\leq 0} \downarrow_0.$$

Analogous to the definition of G , for a homomorphism $\varphi: M \rightarrow K$ between objects $M, K \in \mathcal{O}_0$ we define the corresponding natural transformation $F_\varphi: F_K \rightarrow F_M$ by

$$F_\varphi N := (\varphi^* \otimes \text{Id}_N)^{\leq 0} \downarrow_0 : F_K N \rightarrow F_M N.$$

Hence, fixing $N \in \mathcal{O}_0$, and denoting by $F_- N$ the assignment

$$F_- M: x \mapsto F_x M$$

(x being an object or morphism of \mathcal{O}_0), we see that

$$F_- N = (_ \downarrow_0) \circ (_ {\leq 0}) \circ (_ \otimes N) \circ (_ {}^*).$$

Since $_ {}^*$ is contravariant exact, $_ \otimes N$ is covariant exact, $_ {\leq 0}$ is covariant right exact, and $_ \downarrow_0$ is covariant exact, it follows that $F_- N$ is a contravariant right exact endofunctor on \mathcal{O}_0 , which proves the statement for F . The statement for H follows by duality. \square

Remark 3.6. Note that, since $L(0)^* \cong L(0) \cong {}_{\mathfrak{g}}\mathbb{C}$, with \mathfrak{g} acting trivially on \mathbb{C} , we have isomorphisms

$$\begin{aligned} G_{L(0)} M &= M \otimes L(0) \downarrow_0 \cong M \downarrow_0 = M, \text{ and} \\ F_{L(0)} M &= (M \otimes L(0)^*)^{\leq 0} \downarrow_0 \cong M^{\leq 0} \downarrow_0 = M \end{aligned}$$

natural in M , for any $M \in \mathcal{O}_0$. Hence we have natural isomorphisms

$$G_{L(0)} \cong F_{L(0)} \cong H_{L(0)} \cong \text{Id},$$

where Id denotes the identity functor on \mathcal{O}_0 .

Proposition 3.7. *For any $M \in \mathcal{O}_0$ the following holds.*

- (a) F_M and G_M preserve $\mathcal{F}_0(\Delta)$ and are acyclic on it.
- (b) G_M and H_M preserve $\mathcal{F}_0(\nabla)$ and are acyclic on it.

Proof. G_M preserves $\mathcal{F}_0(\Delta)$ and $\mathcal{F}_0(\nabla)$ by Proposition 2.6. G_M is also acyclic on $\mathcal{F}_0(\Delta)$ and $\mathcal{F}_0(\nabla)$ since G_M is exact.

F_M preserves $\mathcal{F}_0(\Delta)$ by Proposition 2.10. If F_M is acyclic on K and Q , and the sequence

$$0 \rightarrow K \rightarrow N \rightarrow Q \rightarrow 0$$

is exact, then it follows that F_M is acyclic on N . Hence it suffices to show that F_M is acyclic on Verma modules, by induction on the length of Verma flags.

A right exact functor is always acyclic on projective modules, so in particular F_M is acyclic on $\Delta(0)$, since $\Delta(0)$ is projective. Now let

$\lambda \in \mathfrak{h}^*$ with $\lambda < 0$ and $\Delta(\lambda) \in \mathcal{O}_0$, and assume that F_M is acyclic on $\Delta(\mu)$ for all $\mu \in \mathfrak{h}^*$ with $\lambda < \mu$ and $\Delta(\mu) \in \mathcal{O}_0$. All Verma modules fit in a short exact sequence

$$0 \rightarrow K \rightarrow P(\lambda) \rightarrow \Delta(\lambda) \rightarrow 0, \quad (6)$$

where $P(\lambda)$ is projective, and $K \in \mathcal{F}_0(\Delta)$ is filtered by Verma modules $\Delta(\mu)$ with $\lambda < \mu$. In particular, F_M is acyclic on K by the induction hypothesis. Hence, in the induced long exact sequence

$$\cdots \rightarrow \mathcal{L}_{i+1}F_MP(\lambda) \rightarrow \mathcal{L}_{i+1}F_M\Delta(\lambda) \rightarrow \mathcal{L}_iF_MK \rightarrow \cdots$$

we have $\mathcal{L}_{i+1}F_MP(\lambda) = 0$ since $P(\lambda)$ is projective, and $\mathcal{L}_iF_MK = 0$ by the induction hypothesis, so $\mathcal{L}_{i+1}F_M\Delta(\lambda) = 0$ for all $i > 1$. It remains to show that $\mathcal{L}_1F_M\Delta(\lambda) = 0$.

Since $\mathcal{L}_1F_MP(\lambda) = 0$, we have that

$$0 \rightarrow \mathcal{L}_1F_M\Delta(\lambda) \rightarrow F_MK \rightarrow F_MP(\lambda) \rightarrow \Delta(\lambda) \rightarrow 0 \quad (7)$$

is exact. Now, consider the short exact sequence

$$0 \rightarrow M^* \otimes K \rightarrow M^* \otimes P(\lambda) \rightarrow M^* \otimes \Delta(\lambda) \rightarrow 0$$

obtained from (6) by applying the functor $M^* \otimes _$. The modules in the above sequence are all $\mathcal{U}(\mathfrak{n}_-)$ -free, so by Proposition 2.9 we obtain an exact sequence

$$0 \rightarrow F_MK \rightarrow F_MP(\lambda) \rightarrow F_M\Delta(\lambda) \rightarrow 0 \quad (8)$$

by applying $_ \leq^0$, and thus $\mathcal{L}_1F_M\Delta(\lambda) = 0$, by comparing (7) and (8).

Since $H_M = \star \circ F_{M^*} \circ \star$, and \star is a contravariant exact functor swapping $\mathcal{F}_0(\nabla)$ with $\mathcal{F}_0(\Delta)$, it follows by the dual argument to the previous paragraph that H_M preserves $\mathcal{F}_0(\nabla)$ and is acyclic on it. \square

Lemma 3.8. *The functors F , G and H are faithful.*

Proof. Let $M, N \in \mathcal{O}_0$ with a non-zero homomorphism $\varphi: M \rightarrow N$. By the symmetry of the tensor product, we have $G_K \cong G_K_$ for any $K \in \mathcal{O}_0$. In particular, it follows from Remark 3.6 that $G_L(0) = \text{Id}$. Thus $G_\varphi L(0) = (\varphi \otimes \text{Id}_{L(0)}) \downarrow_0 \neq 0$, so G_φ is non-zero and hence G is faithful.

Now, let $m^* \in M^*$, $m^* \neq 0$ be a lowest weight vector of weight $\mu \in \mathfrak{h}^*$ in the image of the map $\varphi^*: N^* \rightarrow M^*$, and let $n^* \in N^*$ with $\varphi^*(n^*) = m^*$. Let $\lambda \in \mathfrak{h}^*$ be the antidominant weight, i.e. with $L(\lambda) = \Delta(\lambda) \in \mathcal{O}_0$, and consider $F_\varphi\Delta(\lambda): F_N\Delta(\lambda) \rightarrow F_M\Delta(\lambda)$. Let $v \in \Delta(\lambda)$ denote a non-zero highest weight vector of $\Delta(\lambda)$.

Since μ is a lowest weight of $\varphi^*(N^*)$ and $N \in \mathcal{O}_0$, it follows that $\lambda + \mu \leq 0$ and $\Delta(\lambda + \mu) \in \mathcal{O}_0$. In particular, by the proof of Proposition 2.9, both $n^* \otimes v$ and $m^* \otimes v$ represent non-zero elements $\overline{n^* \otimes v}$ and $\overline{m^* \otimes v}$ in

$$(N^* \otimes \Delta(\lambda))^{\leq 0} \downarrow_0 = F_N \Delta(\lambda)$$

and

$$(M^* \otimes \Delta(\lambda))^{\leq 0} \downarrow_0 = F_M \Delta(\lambda),$$

respectively. In particular, since

$$F_\varphi \Delta(\lambda) = (\varphi^* \otimes \text{Id}_{\Delta(\lambda)})^{\leq 0} \downarrow_0$$

we see that

$$(F_\varphi \Delta(\lambda))(\overline{n^* \otimes v}) = \overline{\varphi^*(n^*) \otimes v} = \overline{m^* \otimes v} \neq 0.$$

Hence F_φ is non-zero, proving that F is faithful. By duality, it follows that H is faithful. \square

We now conclude the proof of Theorem 3.1 by showing a slightly stronger statement than “ $X_M \cong X_N$ if and only if $M \cong N$ ”.

Proposition 3.9. *Let $M, N \in \mathcal{O}_0$ with $M \not\cong N$. Then*

- (a) $F_M|_{\text{proj}} \not\cong F_N|_{\text{proj}}$,
- (b) $G_M|_{\text{tilt}} \not\cong G_N|_{\text{tilt}}$, and
- (c) $H_M|_{\text{inj}} \not\cong H_N|_{\text{inj}}$,

where $|_{\text{proj}}$, $|_{\text{tilt}}$ and $|_{\text{inj}}$ denote the restrictions to the additive categories of projective, tilting and injective modules, respectively.

Proof. We start by noting that if $G_M \cong G_N$, then

$$M \cong G_M L(0) \cong G_N L(0) \cong N. \quad (9)$$

Assume that $F_M|_{\text{proj}} \cong F_N|_{\text{proj}}$. Since F_M and F_N are right exact, it follows by taking projective presentations that $F_M K \cong F_N K$ for any $K \in \mathcal{O}_0$, i.e. $F_N \cong F_M$. By the uniqueness of right adjoints, this implies that $G_M \cong G_N$ so $M \cong N$ by (9), and hence we have proved part (a). Part (c) follows from (a) by duality (as in the proof of Proposition 3.7).

For part (b), assume that $G_M|_{\text{tilt}} \cong G_N|_{\text{tilt}}$. We recall that each projective module $P \in \mathcal{O}_0$ has a tilting co-resolution, i.e. there are tilting modules $T_0, \dots, T_k \in \mathcal{O}_0$ such that the sequence

$$0 \rightarrow P \rightarrow T_0 \rightarrow \dots \rightarrow T_k \rightarrow 0$$

is exact (for details, see [R, Lemma 6]). Since G_N and G_M are exact and agree on the additive category of tilting modules, this induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \rightarrow & G_M P & \rightarrow & G_M T_0 & \rightarrow & \cdots \rightarrow G_M T_k \rightarrow 0 \\ & & & & \parallel & & \parallel \\ 0 & \rightarrow & G_N P & \rightarrow & G_N T_0 & \rightarrow & \cdots \rightarrow G_N T_k \rightarrow 0 \end{array}$$

By the Five Lemma this induces an isomorphism $G_M P \cong G_N P$, which furthermore is natural, since all isomorphisms in the above diagram are natural. Hence G_M and G_N are naturally equivalent on projective modules, so by the right exactness $G_M \cong G_N$ as in the proof of part (a). By (9) we have $M \cong N$, as required. \square

Proposition 3.10. *For all $M \in \mathcal{F}_0(\Delta)$, $N \in \mathcal{F}_0(\nabla)$ we have that*

- (a) $F_N M$ is projective,
- (b) $G_M N \cong G_N M$ is a tilting module, and
- (c) $H_M N$ is injective.

Proof. For part (a), we need to show that $\text{Hom}(F_N M, _)$ is exact. Since

$$\text{Hom}(F_N M, _) \cong \text{Hom}(M, G_N _),$$

it is equivalent to show that $\text{Hom}(M, G_N _)$ is exact. By Proposition 2.6, $G_N _$ maps any module to a module with a dual Verma flag, since $N \in \mathcal{F}(\nabla)$. Hence, as $G_N _$ is exact, it maps an exact sequence to an exact sequence of modules in $\mathcal{F}(\nabla)$. Finally, $\text{Hom}(M, _)$ is acyclic on $\mathcal{F}(\nabla)$ since $M \in \mathcal{F}(\Delta)$ (see [R, Corollary 2]), so applying $\text{Hom}(M, _)$ to an exact sequence of modules in $\mathcal{F}(\nabla)$ again yields an exact sequence, i.e. $\text{Hom}(M, G_N _)$ is exact.

Part (c) follows from (a) by duality. Finally, part (b) follows directly from Proposition 2.6. \square

Corollary 3.11. *For all $M \in \mathcal{T}_0$, F_M maps tilting modules to projective modules, and H_M maps tilting modules to injective modules.*

In general it is quite difficult to compute $F_M N$ and $H_M N$, but the following is a nice special case.

Proposition 3.12. *For each $\lambda \in \mathfrak{h}^*$ with $\Delta(\lambda) \in \mathcal{O}_0$ we have*

$$\begin{aligned} F_{\nabla(\lambda)} \Delta(\lambda) &\cong \Delta(0), \text{ and} \\ H_{\Delta(\lambda)} \nabla(\lambda) &\cong \nabla(0). \end{aligned}$$

Proof. Let $\mu \in \mathfrak{h}^*$ be such that $\mu < 0$ and $L(\mu) \in \mathcal{O}_0$. Since $\mu < 0$ it follows that $(G_{\nabla(\lambda)} L(\mu))_\lambda = \{0\}$, so

$$\dim \operatorname{Hom}(F_{\nabla(\lambda)} \Delta(\lambda), L(\mu)) \cong \dim \operatorname{Hom}(\Delta(\lambda), G_{\nabla(\lambda)} L(\mu)) = 0.$$

On the other hand, we have

$$\begin{aligned} \dim \operatorname{Hom}(F_{\nabla(\lambda)} \Delta(\lambda), L(0)) &\cong \dim \operatorname{Hom}(\Delta(\lambda), G_{\nabla(\lambda)} L(0)) \\ &\cong \dim \operatorname{Hom}(\Delta(\lambda), \nabla(\lambda)) \\ &= 1, \end{aligned}$$

so $F_{\nabla(\lambda)} \Delta(\lambda)$ has simple top $L(0)$. By Proposition 3.10 $F_{\nabla(\lambda)} \Delta(\lambda)$ is projective, and hence

$$F_{\nabla(\lambda)} \Delta(\lambda) \cong \Delta(0).$$

The second statement follows by duality. \square

Proposition 3.13. *There are natural transformations*

- (a) $G_{\Delta(0)} \twoheadrightarrow \operatorname{Id}, \operatorname{Id} \hookrightarrow G_{\nabla(0)},$
- (b) $\operatorname{Id} \hookrightarrow H_{\Delta(0)}, F_{\nabla(0)} \twoheadrightarrow \operatorname{Id}.$

Proof. Since $F_{L(0)} \cong G_{L(0)} \cong H_{L(0)} \cong \operatorname{Id}$, together with the fact that F is right exact, G is exact and H is left exact, this follows by applying the functors F , G and H to the canonical homomorphisms $\Delta(0) \twoheadrightarrow L(0)$ and $L(0) \hookrightarrow \nabla(0)$. \square

4 (Co-)Monad structures

We briefly recall the definition of a monad and a comonad (sometimes called triple and cotriple, respectively), for details see [M, W]. A monad $(\mathcal{U}, \nabla, \eta)$ on a category \mathcal{C} is an endofunctor $\mathcal{U}: \mathcal{C} \rightarrow \mathcal{C}$ together with two natural transformations $\nabla: \mathcal{U}^2 \rightarrow \mathcal{U}$ and $\eta: \operatorname{Id} \rightarrow \mathcal{U}$ such that the diagrams

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\eta\mathcal{U}} & \mathcal{U}^2 \\ \mathcal{U}\eta \downarrow & \searrow \operatorname{Id} & \downarrow \nabla \\ \mathcal{U}^2 & \xrightarrow{\nabla} & \mathcal{U} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{U}^3 & \xrightarrow{\mathcal{U}\nabla} & \mathcal{U}^2 \\ \nabla\mathcal{U} \downarrow & & \downarrow \nabla \\ \mathcal{U}^2 & \xrightarrow{\nabla} & \mathcal{U} \end{array} \quad (10)$$

commute. Dually, a comonad $(\Omega, \Delta, \varepsilon)$ on a category \mathcal{C} is an endofunctor $\Omega: \mathcal{C} \rightarrow \mathcal{C}$ together with two natural transformations $\Delta: \Omega \rightarrow \Omega^2$

and $\varepsilon: \Omega \rightarrow \text{Id}$ such that the diagrams

$$\begin{array}{ccc}
\Omega & \xleftarrow{\varepsilon\Omega} & \Omega^2 \\
\Omega\varepsilon \uparrow & \swarrow \text{Id} & \uparrow \Delta \\
\Omega^2 & \xleftarrow{\Delta} & \Omega
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\Omega^3 & \xleftarrow{\Omega\Delta} & \Omega^2 \\
\Delta\Omega \uparrow & & \uparrow \Delta \\
\Omega^2 & \xleftarrow{\Delta} & \Omega
\end{array}
\quad (11)$$

commute.

Fix a non-zero highest weight vector v of $\Delta(0)$. Recall that $\mathcal{U}(\mathfrak{g})$ admits a coalgebra structure with counit $\tilde{\varepsilon}: \mathcal{U}(\mathfrak{g}) \rightarrow \mathbb{C}$ and comultiplication $\tilde{\Delta}: \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. This induces two homomorphisms

$$D: \Delta(0) \hookrightarrow \Delta(0) \otimes \Delta(0), uv \mapsto \tilde{\Delta}(u)(v \otimes v), \quad (12)$$

$$E: \Delta(0) \twoheadrightarrow L(0), uv \mapsto \tilde{\varepsilon}(u), \quad (13)$$

for $u \in \mathcal{U}(\mathfrak{n}_-)$, where we identify $L(0)$ with \mathbb{C} via $\bar{v} \mapsto 1$.

Proposition 4.1. *The homomorphisms (12) and (13) induce a comonad $(\Delta(0) \otimes _, \Delta, \varepsilon)$ on $\tilde{\mathcal{O}}$ with Δ injective and ε surjective, and dually a monad $(\nabla(0) \otimes _, \nabla, \eta)$ with ∇ surjective and η injective.*

Proof. Fix $M \in \tilde{\mathcal{O}}$. Applying the functor $_ \otimes M$ to (12) and (13) we obtain the homomorphisms (where as above we identify $L(0)$ with \mathbb{C})

$$\Delta_M := D \otimes \text{Id}_M: \Delta(0) \otimes M \hookrightarrow \Delta(0) \otimes \Delta(0) \otimes M,$$

and

$$\varepsilon_M := E \otimes \text{Id}_M: \Delta(0) \otimes M \twoheadrightarrow M.$$

By the proof of Proposition 2.6, $\Delta(0) \otimes M$ is generated by elements of the form $v \otimes m$, $m \in M$. For such an element, it is trivial to show that

$$((\varepsilon_{\Delta(0) \otimes M}) \circ \Delta_M)(v \otimes m) = ((\text{Id}_{\Delta(0)} \otimes \varepsilon_M) \circ \Delta_M)(v \otimes m) = v \otimes m,$$

and

$$\begin{aligned}
((\Delta_{\Delta(0) \otimes M}) \circ \Delta_M)(v \otimes m) &= ((\text{Id}_{\Delta(0)} \otimes \Delta_M) \circ \Delta_M)(v \otimes m) \\
&= v \otimes v \otimes v \otimes m,
\end{aligned}$$

so the diagrams (11) commute, proving that $(\Delta(0) \otimes _, \Delta, \varepsilon)$ is a comonad on $\tilde{\mathcal{O}}$.

Applying $\star \circ (_ \otimes M^\star)$ to (12) and (13) gives the homomorphisms

$$\nabla_M := (\Delta_M^\star)^\star: \nabla(0) \otimes \nabla(0) \otimes M \twoheadrightarrow \nabla(0) \otimes M, \text{ and}$$

$$\eta_M := (\varepsilon_M^\star)^\star: M \hookrightarrow \nabla(0) \otimes M.$$

By duality, the diagrams (10) commute. \square

We can refine this result to the category \mathcal{O}_0 .

Theorem 4.2. *The homomorphisms (12) and (13) induce a comonad $(G_{\Delta(0)}, \bar{\Delta}, \bar{\varepsilon})$ on \mathcal{O}_0 , with $\bar{\Delta}$ injective and $\bar{\varepsilon}$ surjective, and dually a monad $(G_{\nabla(0)}, \bar{\nabla}, \bar{\eta})$ on \mathcal{O}_0 , with $\bar{\nabla}$ surjective and $\bar{\eta}$ injective.*

We prove Theorem 4.2 in parts, throughout the rest of this section. Define

$$\begin{aligned}\bar{\Delta}_M &: G_{\Delta(0)}M \rightarrow G_{\Delta(0)}G_{\Delta(0)}M, \text{ and} \\ \bar{\varepsilon}_M &: G_{\Delta(0)}M \rightarrow M,\end{aligned}$$

by

$$\begin{aligned}\bar{\Delta}_M &:= \pi_{G_{\Delta(0)}G_{\Delta(0)}M} \circ \Delta_M \circ \iota_{G_{\Delta(0)}M}, \text{ and} \\ \bar{\varepsilon}_M &:= \varepsilon_M \circ \iota_{G_{\Delta(0)}M},\end{aligned}$$

where π_x and ι_x as before denotes natural projections and injections. Let $\bar{\Delta}$ and $\bar{\varepsilon}$ be the natural transformations corresponding to $\bar{\Delta}_M$ and $\bar{\varepsilon}_M$.

Remark 4.3. Similarly as in the case of Remark 3.2, by central character considerations we see that $\pi_{G_{\Delta(0)}G_{\Delta(0)}} \circ \Delta_M$ factors through $\bar{\Delta}_M$, and ε_M factors through $\bar{\varepsilon}_M$, i.e. the diagrams

$$\begin{array}{ccc} \Delta(0) \otimes M & \xrightarrow{\Delta_M} & \Delta(0) \otimes \Delta(0) \otimes M \\ \pi_{G_{\Delta(0)}M} \downarrow & \pi_{G_{\Delta(0)}G_{\Delta(0)}M} \downarrow & \\ G_{\Delta(0)}M & \xrightarrow{\bar{\Delta}_M} & G_{\Delta(0)}G_{\Delta(0)}M \end{array} \quad \begin{array}{ccc} \Delta(0) \otimes M & & \\ \pi_{G_{\Delta(0)}M} \downarrow & \searrow \varepsilon_M & \\ G_{\Delta(0)}M & \xrightarrow{\bar{\varepsilon}_M} & M \end{array}$$

commute.

Lemma 4.4. *The left of the diagrams (11) for the triple $(G_{\Delta(0)}, \bar{\Delta}, \bar{\varepsilon})$ commutes.*

Proof. Fix $M \in \mathcal{O}_0$, with a weight basis $m_1, m_2, \dots \in M$, and consider an element

$$\sum_{i=1}^k (u_i v) \otimes m_i \in G_{\Delta(0)}M,$$

where $u_1, \dots, u_k \in \mathcal{U}(\mathfrak{n}_-)$. Applying Δ_M yields, after collecting the elements of the form $v \otimes _ \otimes _$,

$$\sum_{i=1}^k v \otimes (u_i v) \otimes m_i + \sum_{i,j} (u'_{ij} v) \otimes (u''_{ij} v) \otimes m_i, \quad (14)$$

where $\bar{\varepsilon}(u'_{ij}) = 0$ for all u'_{ij} in the sum on the right. Hence, when applying

$$(\varepsilon \otimes \text{Id}_{G_{\Delta(0)}M}) \circ (\text{Id}_{\Delta(0)} \otimes \pi_{G_{\Delta(0)}M}),$$

the right hand sum of (14) maps to zero, while the left hand sum of (14) maps to

$$\sum_{i=1}^k (u_i v) \otimes m_i.$$

Hence $\bar{\varepsilon}_{G_{\Delta(0)}M} \circ \bar{\Delta}_M = \text{Id}_{G_{\Delta(0)}M}$, so the upper triangle of the left diagram of (11) commutes.

For the lower triangle, consider the following diagram.

$$\begin{array}{ccccc}
\Delta(0) \otimes M & \xrightarrow{\Delta_M} & \Delta(0) \otimes \Delta(0) \otimes M & \xrightarrow{\text{Id}_{\Delta(0)} \otimes \varepsilon_M} & \Delta(0) \otimes M \\
\downarrow \pi_{G_{\Delta(0)}M} & & \downarrow \text{Id}_{\Delta(0)} \otimes \pi_{G_{\Delta(0)}M} & \nearrow \text{Id}_{\Delta(0)} \otimes \bar{\varepsilon}_M & \downarrow \pi_{G_{\Delta(0)}M} \\
& & \Delta(0) \otimes G_{\Delta(0)}M & & \\
& & \downarrow \pi_{G_{\Delta(0)}G_{\Delta(0)}M} & & \\
G_{\Delta(0)}M & \xrightarrow{\bar{\Delta}_M} & G_{\Delta(0)}G_{\Delta(0)}M & \xrightarrow{G_{\Delta(0)}\bar{\varepsilon}_M} & G_{\Delta(0)}M
\end{array}$$

The left square and the triangle commutes by Remark 4.3, and the right quadrangle commutes by Remark 3.2, and hence the diagram commutes. By Proposition 4.1, the top row equals $\text{Id}_{\Delta(0) \otimes M}$, and hence the bottom row equals $\text{Id}_{G_{\Delta(0)}M}$, as required. \square

Corollary 4.5. *The homomorphism $\bar{\varepsilon}_M$ is surjective and the homomorphism $\bar{\Delta}_M$ is injective.*

Proof. Since $\bar{\varepsilon}_M = \varepsilon_M \circ \iota_{G_{\Delta(0)}M}$ it follows that $\bar{\varepsilon}_M$ is surjective as ε_M is surjective. By Lemma 4.4 we have $G_{\Delta(0)}\bar{\varepsilon}_M \circ \bar{\Delta}_M = \text{Id}_{G_{\Delta(0)}M}$, so $\bar{\Delta}_M$ is injective since $\text{Id}_{G_{\Delta(0)}M}$ is injective. \square

Lemma 4.6. *The right of the diagrams (11) for the triple $(G_{\Delta(0)}, \bar{\Delta}, \bar{\varepsilon})$ commutes.*

Proof. We claim that the diagrams

$$\begin{array}{ccccc}
\Delta(0) \otimes M & \xrightarrow{\Delta_M} & \Delta(0) \otimes \Delta(0) \otimes M & \xrightarrow{\text{Id}_{\Delta(0)} \otimes \Delta_M} & \Delta(0) \otimes \Delta(0) \otimes \Delta(0) \otimes M \\
\downarrow \pi_{G_{\Delta(0)}M} & & \downarrow \text{Id}_{\Delta(0)} \otimes \pi_{G_{\Delta(0)}M} & & \downarrow \text{Id}_{\Delta(0)} \otimes \pi_{G_{\Delta(0)}G_{\Delta(0)}M} \\
& & \Delta(0) \otimes G_{\Delta(0)}M & \xrightarrow{\text{Id}_{\Delta(0)} \otimes \bar{\Delta}_M} & \Delta(0) \otimes G_{\Delta(0)}G_{\Delta(0)}M \\
& & \downarrow \pi_{G_{\Delta(0)}G_{\Delta(0)}M} & & \downarrow \pi_{G_{\Delta(0)}G_{\Delta(0)}G_{\Delta(0)}M} \\
G_{\Delta(0)}M & \xrightarrow{\bar{\Delta}_M} & G_{\Delta(0)}G_{\Delta(0)}M & \xrightarrow{G_{\Delta(0)}\bar{\Delta}_M} & G_{\Delta(0)}G_{\Delta(0)}G_{\Delta(0)}M
\end{array}$$

and

$$\begin{array}{ccccc}
\Delta(0) \otimes M & \xrightarrow{\Delta_M} & \Delta(0) \otimes \Delta(0) \otimes M & \xrightarrow{\Delta_{\Delta(0) \otimes M}} & \Delta(0) \otimes \Delta(0) \otimes \Delta(0) \otimes M \\
\downarrow \pi_{G_{\Delta(0)}M} & & \downarrow \text{Id}_{\Delta(0)} \otimes \pi_{G_{\Delta(0)}M} & & \downarrow \text{Id}_{\Delta(0) \otimes \Delta(0)} \otimes \pi_{G_{\Delta(0)}M} \\
& & \Delta(0) \otimes G_{\Delta(0)}M & \xrightarrow{\Delta_{G_{\Delta(0)}M}} & \Delta(0) \otimes \Delta(0) \otimes G_{\Delta(0)}M \\
& & \downarrow \pi_{G_{\Delta(0)}G_{\Delta(0)}M} & & \downarrow \pi_{G_{\Delta(0)}G_{\Delta(0)}G_{\Delta(0)}M} \\
G_{\Delta(0)}M & \xrightarrow{\bar{\Delta}_M} & G_{\Delta(0)}G_{\Delta(0)}M & \xrightarrow{\bar{\Delta}_{G_{\Delta(0)}M}} & G_{\Delta(0)}G_{\Delta(0)}G_{\Delta(0)}M
\end{array}$$

commute. For the first diagram, the left and top right squares commute by Remark 4.3, and the bottom right square commutes by Remark 3.2. For the second diagram, the left and bottom right squares commute by Remark 4.3. For the top right square, we note that

$$\begin{aligned}
\Delta_{\Delta(0) \otimes M} &= D \otimes \text{Id}_{\Delta(0) \otimes M}, \text{ and} \\
\Delta_{G_{\Delta(0)}M} &= D \otimes \text{Id}_{G_{\Delta(0)}M},
\end{aligned}$$

so the square commutes, since

$$\begin{aligned}
D \otimes \pi_{G_{\Delta(0)}M} &= (\text{Id}_{\Delta(0) \otimes \Delta(0)} \otimes \pi_{G_{\Delta(0)}M}) \circ (D \otimes \text{Id}_{\Delta(0) \otimes M}) \\
&= (D \otimes \text{Id}_{G_{\Delta(0)}M}) \circ (\text{Id}_{\Delta(0)} \otimes \pi_{G_{\Delta(0)}M}).
\end{aligned}$$

Thus both diagrams commute. Hence, since

$$(\text{Id}_{\Delta(0)} \otimes \Delta_M) \circ \Delta_M = \Delta_{\Delta(0) \otimes M} \circ \Delta_M$$

by Proposition 4.1, and the fact that projections commute, it follows that

$$G_{\Delta(0)}\bar{\Delta}_M \circ \bar{\Delta}_M = \bar{\Delta}_{G_{\Delta(0)}M} \circ \bar{\Delta}_M,$$

and thus the right of the diagrams (11) commute. \square

From Lemma 4.4 and Lemma 4.6 it follows that $(G_{\Delta(0)}, \bar{\Delta}, \bar{\varepsilon})$ is a comonad on \mathcal{O}_0 , and $\bar{\Delta}$ is injective and $\bar{\varepsilon}$ is surjective by Corollary 4.5. Finally, as in the proof of Proposition 4.1, setting

$$\begin{aligned}\bar{\nabla}_M &:= (\bar{\Delta}_{M^*})^*, \text{ and} \\ \bar{\eta}_M &:= (\bar{\varepsilon}_{M^*})^*,\end{aligned}$$

gives a monad $(G_{\nabla(0)}, \bar{\nabla}, \bar{\eta})$ with $\bar{\nabla}$ surjective and $\bar{\eta}$ injective, by duality, which concludes the proof of Theorem 4.2.

5 Parabolic subcategories

All the previous results can be generalized to the case of the parabolic analogue of \mathcal{O} , in the sense of Rocha-Caridi (see for example [R-C, I]). Let $\mathfrak{p} \subseteq \mathfrak{b}$ be a parabolic subalgebra of \mathfrak{g} , let $\mathfrak{m} \subseteq \mathfrak{n}_-$ with

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p},$$

and let $R_{\mathfrak{m}}$ be the roots of \mathfrak{m} . The parabolic analogies of \mathcal{O} , $\tilde{\mathcal{O}}$, $\mathcal{F}(\Delta)$, etc. are obtained by substituting \mathfrak{n}_- by \mathfrak{m} , \mathfrak{b} by \mathfrak{p} , and R_- by $R_{\mathfrak{m}}$, in the corresponding definition. Thus, for example, $\mathcal{O}^{\mathfrak{p}}$ is defined as the full subcategory of the category of \mathfrak{g} -modules consisting of weight modules that are finitely generated as $\mathcal{U}(\mathfrak{m})$ -modules, and $\mathcal{F}^{\mathfrak{p}}(\Delta)$ is the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ that are free as $\mathcal{U}(\mathfrak{m})$ -modules. Similarly, the partial order \leq on \mathfrak{h}^* is replaced by $\leq_{\mathfrak{p}}$ defined as $\lambda \leq_{\mathfrak{p}} \mu$ if and only if $\lambda - \mu \in \mathbb{N}_0 R_{\mathfrak{m}}$, and so on.

Recall that a generalised Verma module in $\mathcal{O}^{\mathfrak{p}}$ is an element of $\mathcal{F}^{\mathfrak{p}}(\Delta)$ that is generated by a highest weight vector (for details, see [L]). We denote the generalised Verma module generated by a highest weight vector of weight $\lambda \in \mathfrak{h}^*$ by $\Delta^{\mathfrak{p}}(\lambda)$. Furthermore, the objects in $\mathcal{F}^{\mathfrak{p}}(\Delta)$ are precisely the objects in $\mathcal{O}^{\mathfrak{p}}$ that have a generalised Verma filtration.

Almost all statements and proofs of the previous sections hold verbatim with these substitutions. The exception is Proposition 2.9, which needs to be restated in the following (rather complicated) way. Let $\mathfrak{g}^{\mathfrak{p}}$ denote the semisimple part of \mathfrak{p} .

Proposition 5.1. *Let M be a $\mathcal{U}(\mathfrak{m})$ -free module with a $\mathcal{U}(\mathfrak{m})$ -basis*

$$\{v_{ij} \mid i \in I, 1 \leq j \leq k_i\}$$

for some index set I and non-negative integers k_i such that

$$L_i := \mathcal{U}(\mathfrak{g}^{\mathfrak{p}})\{v_{ij} \mid 1 \leq j \leq k_i\}$$

is a k_i -dimensional $\mathfrak{g}^{\mathfrak{p}}$ -module with basis v_{i1}, \dots, v_{ik_i} . Then

$$M^{\leq \lambda} = M^{\leq_{\mathfrak{p}} \lambda} \cong \bigoplus_{\substack{i \in I, \\ L_i \leq \lambda}} \mathcal{U}(\mathfrak{n}_{-})\{v_{i1}, \dots, v_{ik_i}\},$$

where $L_i \leq \lambda$ if $w(v_{ij}) \leq \lambda$ for all $1 \leq j \leq k_i$.

Proof. By completely analogous arguments as in the proof of Proposition 2.9, it follows that

$$M^{\not\leq \lambda} = \sum_{\substack{i \in I, \\ L_i \not\leq \lambda}} \mathcal{U}(\mathfrak{n}_{-})\{v_{i1}, \dots, v_{ik_i}\},$$

and hence the claim follows. \square

All objects of $\mathcal{F}^{\mathfrak{p}}(\Delta)$ satisfy the requirements of Proposition 5.1, and a straightforward argument shows that $M \otimes N^*$ does as well, for all $M \in \mathcal{F}^{\mathfrak{p}}(\Delta)$ and $N \in \mathcal{O}^{\mathfrak{p}}$. In particular, we conclude that the arguments used in Sections 3 and 4 all translate to the parabolic setting.

The main results for the category $\mathcal{O}_0^{\mathfrak{p}}$ are thus the following.

Theorem 5.2. *There exist faithful functors*

$$\begin{aligned} F^{\mathfrak{p}}: \mathcal{O}_0^{\mathfrak{p}} &\hookrightarrow \text{PFun}(\mathcal{O}_0^{\mathfrak{p}})^{op}, M \mapsto F_M^{\mathfrak{p}}, \\ G^{\mathfrak{p}}: \mathcal{O}_0^{\mathfrak{p}} &\hookrightarrow \text{TFun}(\mathcal{O}_0^{\mathfrak{p}}), M \mapsto G_M^{\mathfrak{p}}, \\ H^{\mathfrak{p}}: \mathcal{O}_0^{\mathfrak{p}} &\hookrightarrow \text{IFun}(\mathcal{O}_0^{\mathfrak{p}})^{op}, M \mapsto H_M^{\mathfrak{p}}, \end{aligned}$$

all three satisfying $X_M \cong X_N$ if and only if $M \cong N$ (where $X = F^{\mathfrak{p}}, G^{\mathfrak{p}}, H^{\mathfrak{p}}$).

Proof. These are just the restrictions of F , G , and H to $\mathcal{O}_0^{\mathfrak{p}}$. \square

Proposition 5.3. *For any $M \in \mathcal{O}_0^{\mathfrak{p}}$ the following holds:*

- (a) $F_M^{\mathfrak{p}}$ and $G_M^{\mathfrak{p}}$ preserve $\mathcal{F}_0^{\mathfrak{p}}(\Delta)$ and are acyclic on it.
- (b) $G_M^{\mathfrak{p}}$ and $H_M^{\mathfrak{p}}$ preserve $\mathcal{F}_0^{\mathfrak{p}}(\nabla)$ and are acyclic on it.

Proposition 5.4. *For all $M \in \mathcal{F}_0^{\mathfrak{p}}(\Delta)$, $N \in \mathcal{F}_0^{\mathfrak{p}}(\nabla)$ we have that*

- (a) $F_N^{\mathfrak{p}}M$ is projective,
- (b) $G_M^{\mathfrak{p}}N \cong G_N^{\mathfrak{p}}M$ is a tilting module, and
- (c) $H_M^{\mathfrak{p}}N$ is injective.

Corollary 5.5. *For all $M \in \mathcal{T}_0^{\mathfrak{p}}$, $F_M^{\mathfrak{p}}$ maps tilting modules to projective modules, and $H_M^{\mathfrak{p}}$ maps tilting modules to injective modules.*

Proposition 5.6. *For each $\lambda \in \mathfrak{h}^*$ with $\Delta^{\mathfrak{p}}(\lambda) \in \mathcal{O}_0^{\mathfrak{p}}$ we have*

$$F_{\nabla^{\mathfrak{p}}(\lambda)}^{\mathfrak{p}} \Delta^{\mathfrak{p}}(\lambda) \cong \Delta^{\mathfrak{p}}(0), \text{ and}$$

$$H_{\Delta^{\mathfrak{p}}(\lambda)}^{\mathfrak{p}} \nabla^{\mathfrak{p}}(\lambda) \cong \nabla^{\mathfrak{p}}(0).$$

Proposition 5.7. *The canonical homomorphisms $\Delta^{\mathfrak{p}}(0) \rightarrow L(0)$ and $\Delta^{\mathfrak{p}}(0) \hookrightarrow \Delta^{\mathfrak{p}}(0) \otimes \Delta^{\mathfrak{p}}(0)$ induce a comonad $(G_{\Delta^{\mathfrak{p}}(0)}, \Delta^{\mathfrak{p}}, \varepsilon^{\mathfrak{p}})$ on $\mathcal{O}_0^{\mathfrak{p}}$ with $\Delta^{\mathfrak{p}}$ injective and $\varepsilon^{\mathfrak{p}}$ surjective, and dually a monad $(G_{\nabla^{\mathfrak{p}}(0)}, \nabla^{\mathfrak{p}}, \eta^{\mathfrak{p}})$ with $\nabla^{\mathfrak{p}}$ surjective and $\eta^{\mathfrak{p}}$ injective.*

6 An example: $\mathfrak{sl}_3(\mathbb{C})$

In conclusion we will compute the ‘multiplication table’ given by $G_M N$ and $F_M N$, where M and N run through the simple modules of \mathcal{O}_0 for the algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, see Tables 1 and 2. Let $\alpha, \beta \in \mathfrak{h}^*$ denote the simple roots, let s and t be the corresponding simple reflections (i.e. with $s(\alpha) = -\alpha$ and $t(\beta) = -\beta$), and fix a Weyl-Chevalley basis $X_{\pm\alpha}$, $X_{\pm\beta}$, $X_{\pm(\alpha+\beta)}$, H_{α} , H_{β} .

The ‘dot’ action of the Weyl group $W = S_3$ on \mathfrak{h}^* is defined by

$$w \cdot \lambda := w(\lambda + \rho) - \rho$$

for an element $w \in W$, where $\rho \in \mathfrak{h}^*$ is half the sum of the positive roots. We set $L(w) := L(w \cdot 0)$ for $w \in W$. Let e denote the identity in W . There are two proper parabolic subalgebras, $\mathfrak{p}^{\alpha} := \mathfrak{b} + \langle X_{-\alpha} \rangle_{\mathbb{C}}$ and $\mathfrak{p}^{\beta} := \mathfrak{b} + \langle X_{-\beta} \rangle_{\mathbb{C}}$.

The first row and column for the G -table follow from Remark 3.6. The zero entries are obtained by weight arguments (e.g. (1) and (2)). Similarly one finds that $L(s) \otimes L(s)$ has a highest weight vector of weight $st \cdot 0$. Since $L(s)$ is not $\mathcal{U}(\langle X_{-\beta} \rangle)$ -free, it follows that $G_{L(s)} L(s) \cong L(st)$. By symmetry, $G_{L(t)} L(t) = L(ts)$. Finally, for $G_{L(s)} L(t) \cong G_{L(t)} L(s)$, counting dimensions of the weight spaces shows that $L(st)$ and $L(ts)$ each occur once in the Jordan-Hölder decomposition, and $L(sts)$ occurs twice. Furthermore, since $L(s)$ is $\mathcal{U}(\langle X_{-\alpha} \rangle)$ -free and $L(t)$ is $\mathcal{U}(\langle X_{-\beta} \rangle)$ -free, it follows that $G_{L(s)} L(t)$ is both $\mathcal{U}(\langle X_{-\alpha} \rangle)$ -free and $\mathcal{U}(\langle X_{-\beta} \rangle)$ -free. Hence neither $L(st)$ nor $L(ts)$ can occur in the socle of $G_{L(s)} L(t)$. Finally, we have

$$\left(G_{L(s)} L(t) \right)^{\star} = G_{L(s)^{\star}} L(t)^{\star} = G_{L(s)} L(t),$$

i.e. $G_{L(s)} L(t)$ is self-dual, so neither $L(st)$ nor $L(ts)$ can occur in the top of $G_{L(s)} L(t)$. We conclude that the Loewy series of $G_{L(s)} L(t)$ is

$$G_{L(s)} L(t) \cong \begin{matrix} L(sts) \\ L(st) \ L(ts) \\ L(sts) \end{matrix}.$$

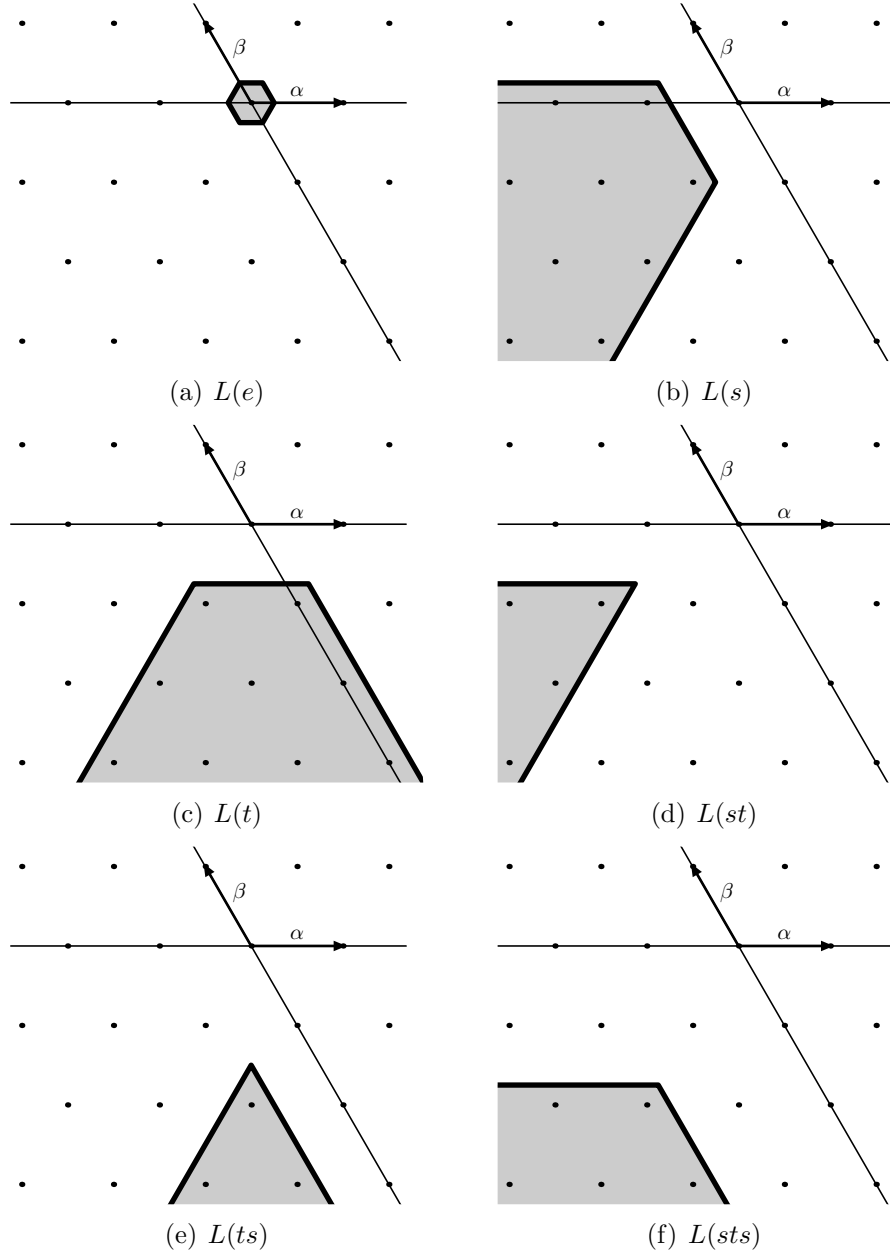


Figure 1: The simple modules in \mathcal{O}_0 for the algebra $\mathfrak{sl}_3(\mathbb{C})$. Each dot is an integral weight, and the grey areas show the support of the corresponding module. Each non-empty weight space has dimension 1 except for $L(sts)$, for which the dimensions are given by Kostant's function.

$G_M N$	$L(e)$	$L(s)$	$L(t)$	$L(st)$	$L(ts)$	$L(sts)$
$L(e)$	$L(e)$	$L(s)$	$L(t)$	$L(st)$	$L(ts)$	$L(sts)$
$L(s)$	$L(s)$	$L(st)$	$\begin{smallmatrix} L(sts) \\ L(st) L(ts) \\ L(sts) \end{smallmatrix}$	0	$L(sts)$	0
$L(t)$	$L(t)$	$\begin{smallmatrix} L(sts) \\ L(st) L(ts) \\ L(sts) \end{smallmatrix}$	$L(ts)$	$L(sts)$	0	0
$L(st)$	$L(st)$	0	$L(sts)$	0	0	0
$L(ts)$	$L(ts)$	$L(sts)$	0	0	0	0
$L(sts)$	$L(sts)$	0	0	0	0	0

Table 1: The “multiplication table” for the bifunctor G on the simple modules in \mathcal{O}_0 for $\mathfrak{sl}_3(\mathbb{C})$.

$F_M N$	$L(e)$	$L(s)$	$L(t)$	$L(st)$	$L(ts)$	$L(sts)$
$L(e)$	$L(e)$	$L(s)$	$L(t)$	$L(st)$	$L(ts)$	$\Delta(sts)$
$L(s)$	0	$L(e)$	0	$\Delta^{\mathfrak{p}^\beta}(s)$	0	$\Delta(ts) \oplus P(t)$
$L(t)$	0	0	$L(e)$	0	$\Delta^{\mathfrak{p}^\alpha}(t)$	$\Delta(st) \oplus P(s)$
$L(st)$	0	0	0	$\Delta^{\mathfrak{p}^\beta}(e)$	0	$\Delta(t)$
$L(ts)$	0	0	0	0	$\Delta^{\mathfrak{p}^\alpha}(e)$	$\Delta(s)$
$L(sts)$	0	0	0	0	0	$\Delta(e)$

Table 2: The “multiplication table” for the bifunctor F on the simple modules in \mathcal{O}_0 for $\mathfrak{sl}_3(\mathbb{C})$.

The corresponding table for F is given in Table 2. Since $F_{L(0)}M = M$, the first row is immediate. Furthermore, by Proposition 3.12, and the fact that $L(sts) = \Delta(sts) = \nabla(sts)$ we have $F_{L(sts)}L(sts) = \Delta(e)$. Similarly, by Proposition 5.6 and the fact that $L(st) = \Delta^{\mathfrak{p}^\beta}(st) = \nabla^{\mathfrak{p}^\beta}(st)$ we have $F_{L(st)}L(st) = \Delta^{\mathfrak{p}^\beta}(e)$ (and similarly for $F_{L(ts)}L(ts)$). Using the adjointness of F and G , we can easily determine the top of $F_{L(i)}L(j)$, i.e. $L(k)$ is in the top of $F_{L(i)}L(j)$ if and only if $L(j)$ is in the socle of $G_{L(i)}L(k)$. In particular, this fact and the G -table gives us all the 0's in the table.

The remaining cases need some additional case by case arguments. We begin with $F_{L(s)}L(s)$. By adjointness, Table 1 shows that $F_{L(s)}L(s)$ has a simple top $L(e)$. Since $L(s) \in \mathcal{O}_0^\beta$, it follows that the possible modules are $L(e)$ and

$$\Delta^{\mathfrak{p}^\beta}(e) = \frac{L(e)}{L(s)}.$$

But by Proposition 5.3 we have $G_{L(s)}\Delta^{\mathfrak{p}^\beta}(e) \in \mathcal{F}_0^\beta(\Delta)$, so by analysing the weights we see that

$$G_{L(s)}\Delta^{\mathfrak{p}^\beta}(e) = \Delta^{\mathfrak{p}^\beta}(s) = \frac{L(s)}{L(st)}.$$

Hence

$$\begin{aligned} \dim \operatorname{Hom}_{\mathfrak{g}}(F_{L(s)}L(s), \Delta^{\mathfrak{p}^\beta}(e)) &= \dim \operatorname{Hom}_{\mathfrak{g}}(L(s), G_{L(s)}\Delta^{\mathfrak{p}^\beta}(e)) \\ &= \dim \operatorname{Hom}_{\mathfrak{g}}(L(s), \Delta^{\mathfrak{p}^\beta}(s)) \\ &= 0, \end{aligned}$$

and so $F_{L(s)}L(s) \neq \Delta^{\mathfrak{p}^\beta}(e)$ and we conclude that $F_{L(s)}L(s) = L(e)$. Analogously, we get $F_{L(t)}L(t) = L(e)$.

Since $L(sts) = \Delta(sts)$, we have $F_{L(st)}L(sts) \in \mathcal{F}_0(\Delta)$ by Proposition 3.7, and by the proof of Proposition 2.10 we know that the Verma modules $\Delta(\lambda)$ occuring in the Verma flag of $F_{L(st)}L(sts)$ are the ones satisfying $\lambda \in sts \cdot 0 - \operatorname{Supp} L(st)$ and $\lambda \leq 0$, with multiplicity equal to the dimension of the weight space of $L(st)$ of weight $sts \cdot 0 - \lambda$. The only such weight is $t \cdot 0$, with multiplicity 1. Hence, $F_{L(st)}L(sts) = \Delta(t)$. Analogously, $F_{L(ts)}L(sts) = \Delta(s)$.

Since $L(st) = \Delta^{\mathfrak{p}^\beta}(st)$, we have $F_{L(st)}L(st) \in \mathcal{F}_0^\beta(\Delta)$. By a similar analysis as for $F_{L(st)}L(sts)$, using the proof of Proposition 5.1, we find that $F_{L(s)}L(st)$ has only one generalised Verma quotient, $\Delta^{\mathfrak{p}^\beta}(s)$, so $F_{L(s)}L(st) = \Delta^{\mathfrak{p}^\beta}(s)$. Similarly, $F_{L(t)}L(ts) = \Delta^{\mathfrak{p}^\alpha}(t)$.

Finally, for $F_{L(s)}L(sts)$, by the same analysis as for $F_{L(st)}L(sts)$ we have that $F_{L(s)}L(sts)$ has a Verma flag with Verma quotients $\Delta(e)$, $\Delta(t)$ and $\Delta(ts)$, each with multiplicity 1. Furthermore, using adjointness we find from Table 1 that $F_{L(s)}L(sts)$ has top $L(ts) \oplus L(t)$. Thus,

$F_{L(s)}L(sts)$ is a quotient of $P(ts) \oplus P(t)$. The module $P(ts) \oplus P(t)$ has the following standard filtration:

$$P(ts) \oplus P(t) = \begin{matrix} \Delta(ts) \\ \Delta(t) \Delta(s) \oplus \Delta(t) \\ \Delta(e) \end{matrix}.$$

It is easy to see that this implies that

$$F_{L(s)}L(sts) = \Delta(ts) \oplus P(t).$$

By symmetry, $F_{L(t)}L(sts) = \Delta(st) \oplus P(s)$, which completes the table.

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